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A CENSUS OF CUBIC FOURFOLDS OVER $\mathbb{F}_2$

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Abstract. We compute a complete set of isomorphism classes of cubic fourfolds over $\mathbb{F}_2$. Using this, we are able to compile statistics about various invariants of cubic fourfolds, including their counts of points, lines, and planes; all zeta functions of the smooth cubic fourfolds over $\mathbb{F}_2$; and their Newton polygons. One particular outcome is the number of smooth cubic fourfolds over $\mathbb{F}_2$, which we fit into the asymptotic framework of discriminant complements. Another motivation is the realization problem for zeta functions of $K3$ surfaces. We present a refinement to the standard method of orbit enumeration that leverages filtrations and gives a significant speedup. In the case of cubic fourfolds, the relevant filtration is determined by Waring representation and the method brings the problem into the computationally tractable range.

Introduction

The study of cubic fourfolds over finite fields (e.g., [1], [2], [11], [14], etc.) has grown as a respectable side industry to the main threads of investigation for cubic fourfolds over the complex numbers, including the rationality problem and its connections to derived categories, algebraic cycles, K3 surfaces, and hyperkähler varieties. In this paper and its accompanying code [4], we generate a database of all cubic fourfolds over $\mathbb{F}_2$ up to isomorphism. We also compute many of their most important invariants, including their automorphism groups, their point counts, and information about their algebraic cycles. In particular, we can report the following.

Theorem 1. *Of the* $3\,718\,649$ *isomorphism classes of cubic fourfolds over* $\mathbb{F}_2$*, exactly* $1\,069\,562$ *are smooth, of which* $533\,262$ *are ordinary,* 8688 *are supersingular,* $107\,552$ *are Noether–Lefschetz general, and* $702\,153$ *contain a plane. The smooth cubic fourfolds admit* $86\,472$ *distinct zeta functions.*

The algorithmic methods to generate our database of cubic fourfolds are of independent interest: we present a new technique for enumerating a complete set of orbit representatives of a finite group G acting linearly on a high-dimensional vector space V over a finite field that leverages G-stable filtrations of V. In the case of cubic fourfolds over $\mathbb{F}_2$, the relevant action is the representation of $G = \mathrm{GL}_6(\mathbb{F}_2)$ on the 56-dimensional $\mathbb{F}_2$-vector space $V = \mathrm{Sym}^3(\mathbb{F}_2^6)$ of homogeneous cubic forms in

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six variables. In certain situations, our method provides a substantial speedup over naive orbit partition algorithms. The advantage of our method, assuming the existence of good G-stable filtrations, is that we do not need to iterate through every element of V. A complexity analysis in §1.2 shows that under favorable situations our method is linear in the number of orbits, which is asymptotically optimal; in the case of cubic fourfolds over $\mathbb{F}_2$, our method gives a roughly square-root speedup.

Our work on cubic fourfolds was partially inspired by Kedlaya and Sutherland's census [33] of quartic $K3$ surfaces over $\mathbb{F}_2$. There, a complete partition of quartic surfaces into $\mathrm{GL}_4(\mathbb{F}_2)$-orbits was achieved in a few days on a powerful computer; with our method, it takes 3 minutes on a laptop to compute a complete set of orbit representatives. They also compute the zeta functions of the smooth orbits, as well as a longer list of potential zeta functions of $K3$ surfaces over $\mathbb{F}_2$. This is achieved by enumerating the candidate Weil polynomials on the middle ℓ-adic cohomology [33, Computation 3(c)]. Kedlaya and Sutherland pose the following.

Problem 1. *Determine the set of zeta functions of $K3$ surfaces defined over $\mathbb{F}_2$.*

We remark that the Tate conjecture for $K3$ surfaces (proved by [42], [9], [10], [39], [34], [40], [32]) implies that there are finitely many isomorphism classes of $K3$ surfaces defined over a fixed finite field by the work of Lieblich, Maulik, and Snowden [38], which holds in any characteristic. A resolution of Problem 1 would provide a kind of Honda–Tate theory for $K3$ surfaces. The work of Taelman [47] implies that the transcendental part of every Weil polynomial in [33, Computation 3(c)] is expected to arise from some $K3$ surface defined over a *suitable extension* of the base field, but we are interested in which zeta functions arise from $K3$ surfaces over $\mathbb{F}_2$.

Over the complex numbers, cubic fourfolds are Fano varieties of $K3$ *type*, with the Hodge structures on their middle cohomology resembling those of $K3$ surfaces. Hassett [25] classified those cubic fourfolds that admit Hodge-theoretically *associated K3 surfaces*, namely the *admissible* special cubic fourfolds. Over a finite field, the Weil polynomial on the middle dimensional ℓ-adic cohomology of a special cubic fourfold has a factor (the nonspecial Weil polynomial) that looks like the Weil polynomial of a $K3$ surface, and we would expect this polynomial to be realizable by a $K3$ surface defined over $\mathbb{F}_2$ whenever a Hodge-theoretically associated $K3$ surface is defined over $\mathbb{F}_2$. Thus our computation of the zeta functions of cubic fourfolds (see §4) provides many new Weil polynomials that should arise from $K3$ surfaces, and fertile ground for the arithmetic study of the associated $K3$ surface. In cases where there is an explicit algebraic construction of an associated $K3$ surface, for example, for cubic fourfolds containing a plane, the nonspecial Weil polynomial of the cubic is the primitive Weil polynomial of some $K3$ surface over $\mathbb{F}_2$. On the other hand, our census exhibits many explicit special cubic fourfolds that cannot have an associated K3 surface because such a K3 would have "negative point counts" as well as certain special cubic fourfolds that are not expected to have associated $K3$ surfaces over $\overline{\mathbb{F}}_2$, yet whose nonspecial Weil polynomial is still contained on Kedlaya and Sutherland's list, raising further questions about associated $K3$ surfaces over finite fields (see the forthcoming work of the first and third authors [3]). Future census projects could help further populate the list of Weil polynomials that are realized by $K3$ surfaces over $\mathbb{F}_2$.

Finally, as stated in Theorem 1, our census also provides a count of the $\mathbb{F}_2$-points of the complement of the generic discriminant of cubic forms in six variables. From this, we find that the probability that a random cubic fourfold is smooth is about 29%, and we connect this to asymptotic results of Poonen [44], Church–Ellenberg–Farb [12], Vakil–Wood [49], and Howe [29] in algebraic geometry, number theory, and topology.

This article is organized as follows. In §1, we present our new method for computing orbit representatives for a finite group G acting on a finite vector space V admitting a filtration by G-stable subspaces, which we coin the "filtration method." We also compare the computational complexity of our method compared with that of more traditional methods. In §2, we describe the range of applicability of the filtration method to enumerating degree d hypersurfaces in $\mathbb{P}^n$ over $\mathbb{F}_q$, including the case of cubic fourfolds over $\mathbb{F}_2$. Finally, in §3 and §4, we compute many invariants associated to cubic fourfold, including their counts of points, lines, and planes, their automorphism groups, and their zeta functions. We also discuss many connections and complements to the existing literature.

1. Orbits via filtrations

Let k be a finite field and V be a finite-dimensional k-vector space on which a finite group G acts linearly and faithfully. For $v \in V$ we denote by $G.v$ the G-orbit containing v, and by G_v the stabilizer subgroup of v. If we are only interested in the cardinality of the orbit set V/G, then we can use the orbit counting formula (sometimes called "Burnside's Lemma" or the "Cauchy–Frobenius Lemma")

$$|V/G| = \frac{1}{|G|}\sum_{g\in G}|V^g| = \frac{|C|}{|G|}\sum_{c\in C}|E_1(c)|, \tag{1}$$

where C is the set of conjugacy classes in G and $E_1(c)$ is the 1-eigenspace of a representative of $c \in C$, whose cardinality does not depend on the representative.

However, to assemble a list of orbit representatives, one has to work harder. To do this, one typically runs a *naive orbit partition algorithm*, sometimes called *union-find* (see §1.2 for more details), to sort each element of V into orbits under G, as is done for quartic surfaces over $\mathbb{F}_2$ in [33]. Alternatively, one could develop a sufficiently good G-invariant hash function on V and try randomly sampling elements of V until one finds elements in all the orbits. The random sampling method will often succeed in identifying elements in all large orbits after sampling $O\big(|V/G|\log(|V/G|)\big)$ elements, but it can fail to find elements in small orbits in reasonable time. This method has been used successfully by Costa, Harvey, and Kedlaya (as reported by Costa [13]) to give a census of quartic $K3$ surfaces over $\mathbb{F}_3$ and was used by Halleck-Dubé [24] to enumerate a set of orbit representatives for 99.9% of the cubic fourfolds over $\mathbb{F}_2$.

However, working directly on V may prove to be too costly (as in the case of cubic fourfolds over $\mathbb{F}_2$), and we introduce a method that can avoid this.

1.1. **Filtration method.** Suppose that there is a filtration of the G-module V

$$0 = W_0 \subset \cdots \subset W_\ell \subset V$$

by G-submodules $W_i \subset V$, such that enumerating G-orbits in all of the associated graded pieces W_{i+1}/W_i becomes a feasible task. Then using such a filtration, we are able to compute a full set of G-orbit representatives for V by chasing lifts of G-orbits of V/W_ℓ up the successive quotients

$$V \to V/W_1 \to \cdots \to V/W_\ell.$$

Let us illustrate the method with a single-step filtration

$$0 \subset W \subset V$$

and consider $U = V/W$ with G-equivariant quotient map $\pi : V \to U$. We first note that every G-orbit $G.v$ in V/G maps to a G-orbit $G.\pi(v)$ in U/G. This lets us write the orbit set V/G as a disjoint union

$$V/G = \bigsqcup_{O \in U/G} \pi^{-1}(O)/G \tag{2}$$

over the orbits $O \in U/G$.

The following elementary lemma, which is easily checked, shows that a complete set of orbit representatives for G acting on $\pi^{-1}(O)$ can be obtained by considering the action of a smaller group on a smaller subset; this holds in the more general context of finite G-sets, i.e., finite sets with the action of a group G.

Lemma 1.1. *Let X and Y be finite G-sets and $\pi : X \to Y$ a G-equivariant map. Let $x \in X$ and $y = \pi(x)$. Then:*

1. *The fiber $\pi^{-1}(y)$ is a G_y-set.*
2. *Let $O = G.y \subset Y$ denote the G-orbit of y. Then the map*
$$\pi^{-1}(y)/G_y \to \pi^{-1}(O)/G$$
defined by $G_y.x \mapsto G.x$, is a bijection.
3. $G_x \leq G_y$.

The upshot of Lemma 1.1(2) is that computing a set of orbit representatives for $\pi^{-1}(y)/G_y$ is less expensive that for $\pi^{-1}(O)/G$.

Remark 1.2. In the context of computational group theory, the fibers of a G-equivariant map of G-sets $X \to Y$ form what is known as a block system for the action of G on X, see [28, §2.2.5, §4.3]. The algorithm developed here could be adapted to the more general context of group actions on sets in the presence of a filtration of block systems.

In our context, $X = V$, $Y = U$, $\pi : V \to U$ is the natural quotient map. Then (2) and Lemma 1.1(2) show that

$$V/G = \bigsqcup_{G.(v+W) \in U/G} (v + W)/G_{\pi(v)}, \tag{3}$$

where the disjoint union is taken over a set of coset representatives $v \in V$ of orbit representatives of G acting on U.

If the computation of a set of orbit representatives (and their stabilizers) for G acting on U is still intractible with a generic orbit-finding algorithm, we can apply the same technique to a G-invariant subspace $U' \subset U$, where $U' = W'/W$ where $0 \subset W \subset W' \subset V$ is a filtration of G-invariant subspaces. This way, we can recursively leverage a filtration of V by G-invariant subspaces.

We now give the full description of an algorithm that uses this principle with successive quotients to find a set of orbit representatives of G acting on V.

Algorithm 1.3. `Orbits`$(G, X, \mathcal{F})$
Input:

- A k-vector space V with the action of a group G.
- A known G-invariant filtration $\mathcal{F}\colon 0 = W_0 \subset \cdots \subset W_\ell \subset V$ of length ℓ.
- A G-invariant affine subspace X of V such that $X + W_\ell = X$, i.e., X is a union of cosets for W_ℓ.

Output: A complete set of orbit representatives for G acting on X, together with their stabilizers.

Steps:

(1) If $\ell = 0$ then **return** `Orbits`(G, X), a set of orbit representatives of G acting on X together with their stabilizers.
(2) Set $\overline{\mathcal{F}} : 0 = W_1/W_1 \subset \cdots \subset W_\ell/W_1 \subset V/W_1$, a G-invariant filtration of length $\ell - 1$, i.e., $\overline{\mathcal{F}}$ is the reduction of $\mathcal{F}$ modulo W_1. Let $\pi_1 : V \to V/W_1$ be the natural quotient map.
(3) Compute `Orbits`$(G, X/W_1, \overline{\mathcal{F}})$ via recursion.
(4) For each orbit representative $y \in X/W_1$ with stabilizer G_y found in the previous step, compute `Orbits`$(G_y, \pi_1^{-1}(y))$ together with their stabilizers.
(5) **return** the union of results from step (4).

The orbit computations in Steps (1) and (4) above can be computed by a generic algorithm. Our implementation uses the default methods in `Magma` [8].

Our main application is when $X = V$ is the k-vector space of homogeneous polynomials of degree d in n variables and $G = \mathrm{GL}_n(k)$, however it could be useful (e.g., in the in context of step (4) of the algorithm) to work with X an arbitrary union of cosets for a G-invariant subspace, see [50, §1.2]. Also see Remark 1.2 for other potential generalizations.

1.2. **Complexity comparison.** We consider the situation of a finite group G acting on a finite set X and define the expected stabilizer order

$$e_G(X) = \frac{1}{|X|} \sum_{x \in X} |G_x|$$

of the G-set X. Then $1 \leq e_G(X) \leq |G|$ with $e_G(X) = 1$ if and only if G acts freely on X and $e_G(X) = |G|$ if and only if G acts trivially on X. We remark that the proof of the orbit counting formula implies that

$$|X/G| = e_G(X) \frac{|X|}{|G|}.$$

A naive orbit partition algorithm, also known as union11-find, to partition all the elements of X into orbits, works by iteratively selecting the next unlabelled element $x \in X$ and then by labelling the elements of $G.x$ as being in the same orbit by enumerating over G. One easily finds the runtime of this procedure.

Lemma 1.4. *The runtime of a naive orbit partition algorithm to partition the elements of X into orbits under G is proportional to*

$$|G| \cdot |X/G| = e_G(X) \cdot |X|.$$

In the situation we are interested in, where X is a vector space with a faithful G-representation, we usually have that $e_G(X)$ is approximately equal to 1. For example, the expected order of the stabilizer of a cubic fourfold over $\mathbb{F}_2$ turns out to be approximately 1.04.

The main improvement introduced by using the filtration method is that one runs several orbit partitions over linear spaces of smaller dimension. We give a precise estimate of the improvement in runtime.

Lemma 1.5. *For a single-step filtration $0 \subset W \subset V$ of G-modules, with $U = V/W$, the runtime complexity of Algorithm 1.3 is proportional to*

$$\frac{e_G(W)e_G(U)}{e_G(V)} \cdot |V/G|.$$

Proof. Let $\pi : V \to U$ be the quotient map. By (3), the runtime complexity is proportional to

$$\sum_v |G_{\pi(v)}| \cdot |(v+W)/G_{\pi(v)}|, \tag{4}$$

where the sum is over a set of orbit representatives $v \in V$ of orbit representatives of G acting on U. One immediately sees that this is bounded by $|G| \cdot |V/G| = e_G(V) \cdot |V|$, which is the runtime of a naive orbit partition algorithm, see Lemma 1.4. On the other hand, for each v we have

$$|G_{\pi(v)}| \cdot |(v+W)/G_{\pi(v)}| = \sum_{g \in G_{\pi(v)}} |(v+W)^g| \leq \sum_{g \in G} |(v+W)^g| \leq \sum_{g \in G} |W^g| = |G| \cdot |W/G|,$$

where the rightmost inequality follows from the observation that for any element $z \in (v+W)^g$, translation by z induces a bijection $W^g \to (v+W)^g$. Thus (4) is bounded by $|W/G| \cdot |U/G| \cdot |G| = e_G(W)e_G(U)/e_G(V) \cdot |V/G|$. □

Lemma 1.5 shows that the filtration method, in the presence of a nontrivial filtration, will strictly improve upon (unless the action is trivial) a naive orbit partition for the purposes of finding a set of orbit representatives.

If we make the heuristic assumption that the expected stabilizer orders of W, U, and V are all approximately equal to 1, then the asymptotic runtime is linear in the total number of orbits in V, i.e., linear in the size of the output. Since each orbit needs to be visited at least once by any algorithm (in particular, to write a representative), a runtime that is linear in the number of orbits is a constant multiple of the best possible. This assumption seems to hold in the cases identified in §2.4.

2. Enumerating hypersurfaces

Let n, d be positive integers, and $\mathbb{F}_q$ denote the finite field with q elements. In this section, we explain when and how the filtration method (Algorithm 1.3) can be used to produce a complete enumeration of the set of $\mathbb{F}_q$-isomorphism classes of degree d hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{F}_q}$.

Proposition 2.1. *Let k be any field. If $n \geq 3$ and $d \geq 3$, then the set of k-isomorphism classes of degree d hypersurfaces in $\mathbb{P}^{n+1}_k$ is in bijection with the set of $\mathrm{PGL}_{n+2}(k)$-orbits on the set of lines $\mathbb{P}(\mathrm{Sym}^d(k^{n+2}))(k)$ in $\mathrm{Sym}^d(k^{n+2})$.*

Moreover, if every element of $k^\times$ is a dth power, e.g., if $k = \mathbb{F}_q$ and $q-1$ is relatively prime to d, then the set of k-isomorphism classes of degree d hypersurfaces in $\mathbb{P}^{n+1}_k$ is in bijection with the set of nonzero $\mathrm{GL}_{n+2}(k)$-orbits on the k-vector space $\mathrm{Sym}^d(k^{n+2})$.

Proof. The Grothendieck–Lefschetz Theorem [23, Exp. XII, Corollaire 3.6] says that any automorphism of a (not necessarily smooth) hypersurface of dimension $n \geq 3$ and degree $d \geq 3$ extends to the ambient $\mathbb{P}_k^{n+1}$. In particular, two such hypersurfaces are k-isomorphic if and only if they lie in the same $\mathrm{PGL}_{n+2}(k)$-orbit of the linear system of $\mathcal{O}_{\mathbb{P}^{n+1}}(d)$.

When every element of $k^\times$ is a dth power, the central $\mathbb{G}_m \subset \mathrm{GL}_{n+2}$ acts transitively on the set of multiples of a given homogeneous form of degree d over k, so that the natural surjective map

$$\mathrm{Sym}^d(k^{n+2})/\mathrm{GL}_{n+2}(k) \to \mathbb{P}(\mathrm{Sym}^d(k^{n+2}))(k)/\mathrm{PGL}_{n+2}(k)$$

is a bijection. □

2.1. **A filtration on cubic fourfolds over $\mathbb{F}_2$.** By Proposition 2.1, the $\mathrm{GL}_6(\mathbb{F}_2)$-orbits of nonzero cubic forms in 6 variables are precisely the $\mathbb{F}_2$-isomorphism classes of cubic fourfolds. We will now show how the filtration method lets us enumerate a representative cubic fourfold in each isomorphism class, equivalently, a complete set of nonzero $\mathrm{GL}_6(\mathbb{F}_2)$-orbit representatives on $V = \mathrm{Sym}^3(\mathbb{F}_2^6)$.

Using Equation (1), one computes that that the number of orbits is 3 718 650, which seems manageable compared the total number $|V| = 2^{56}$ of cubics. Even with the unrealistically generous assumption that computing f^g, for some general $f \in V$ and $g \in \mathrm{GL}_6(\mathbb{F}_2)$, takes 10^{-9} (s), a naive orbit partition algorithm applied to V using a single 4 GHz processor would take at least

$$\frac{2^{56}}{86400 \cdot 4 \cdot 10^9} \sim 208 \text{ days}.$$

In other words, a direct, parallelized, and highly optimized implementation of a naive orbit partition algorithm might enumerate all of the orbits, but it would nevertheless take a while.

Instead, we make use of a natural two-step filtration of G-submodules

$$0 \subset W_1 \subset W_2 \subset V$$

and employ Algorithm 1.3. Here, $W_1 \subset V$ is the subspace of Waring representable cubic forms, i.e., those that can be written as a sum of cubes of linear forms, and $W_2 \subset V$ is the subspace of cubic forms that can be written as a sum of products of a linear form and a square of a linear form. In other words,

$$W_1 = \mathrm{span}\{l^3 \;:\; l \in \mathrm{Sym}^1(\mathbb{F}_2^6)\}$$
$$W_2 = \mathrm{span}\{l_1 \cdot l_2^2 \;:\; l_1, l_2 \in \mathrm{Sym}^1(\mathbb{F}_2^6)\}.$$

A computer calculation shows that $\dim_{\mathbb{F}_2}(W_1) = 21$ and $\dim_{\mathbb{F}_2}(W_2) = 36$.

Our implementation in `Magma` [8] of Algorithm 1.3 with this particular two-step filtration outputs a complete set of representatives for the 3 718 650 orbits in V with runtime under 100 minutes on a household laptop computer. The number of orbits we found matches the number of orbits produced by Burnside's formula, which is a nice sanity-check for the correctness of the algorithm.

A complete set of orbit representatives together with the code to read them is available as an ancillary file in the arXiv distribution of this article. Our complete code library, together with data and various sanity checks, is available from [4]. The intrinsic `LoadCubicOrbitData` from the `CubicLib.m` library reads in the orbit representatives.

2.2. **A filtration on quartic surfaces over $\mathbb{F}_2$.** As an alternative to the orbit partition method employed in [33], we implemented the filtration method to find a representative of each $\mathrm{PGL}_4(\mathbb{F}_2)$-orbit of quartic surfaces over $\mathbb{F}_2$. We note that because there are automorphisms of $K3$ surfaces that do not fix a given degree 4 polarization, some isomorphism classes split up into different linear orbits, but a list of orbit representatives will contain a complete set of isomorphism classes as a subset. By Lemma 2.1, we can compute the $\mathrm{GL}_4(\mathbb{F}_2)$-orbits on the 35-dimensional $\mathbb{F}_2$-vector space $V = \mathrm{Sym}^4(\mathbb{F}_2^4)$ of homogeneous quartic forms in four variables. Let $W \subset V$ be the submodule spanned by all quartics of the form $l_1^3 l_2 + l_1 l_2^3$ where l_1 and l_2 are linear forms. A computer calculation shows that $\dim_{\mathbb{F}_2}(W) = 20$. Then Algorithm 1.3, with the one-step filtration $0 \subset W \subset V$, finds a complete set of orbit representatives for the 1 732 564 orbit in V in about 3 minutes on a laptop computer. We stumbled upon the G-submodule $W \subset V$ using `Magma`'s `IsIrreducible` intrinsic.

2.3. **Cubic fourfolds over $\mathbb{F}_3$.** Unfortunately, the $\mathrm{GL}_6(\mathbb{F}_3)$-module of the 56-dimensional $\mathbb{F}_3$-vector space V of cubic forms in six variables has an irreducible composition factor of dimension 50. Hence the filtration method alone does not provide a sufficiently significant speedup to make the orbit enumeration problem computational tractable in this case.

2.4. **Filtration method for general hypersurfaces.** One may wonder about the extent to which the filtration method presented in §1 can aid in the census of isomorphism classes of hypersurfaces of dimension n and degree d over $\mathbb{F}_q$ for various (n, d, q). Since the dth symmetric power of the standard representation of the linear algebraic group GL_{n+2} is irreducible, the $\mathrm{GL}_{n+2}(\mathbb{F}_q)$-representation $\mathrm{Sym}^d(\mathbb{F}_q^{n+2})$ is irreducible for all $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) > d$ (cf. [46, Theorem 1.1]), hence the filtration method is not applicable. On the other hand, the $\mathrm{GL}_{n+2}(\mathbb{F}_q)$-representation $\mathrm{Sym}^d(\mathbb{F}_q^{n+2})$ is reducible for all $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \leq d$, hence the filtration method does offer some speedup when q is small. However, one quickly sees (see Tables 1, 2, 3) even for moderately small parameters that such a census is infeasible simply because it is not computationally practical to store the answer. (We consider 10^{14} a generous allowance for the maximum number of orbits that can be computed.)

When the total number of orbits is reasonably sized, the determination of whether a particular value of (n, d, q) is in the feasible range is based on timings for computing $g.x$ on a standard laptop. We also assume that 100 cores are available to parallelize the computation over the course of a year. Thus, the projections listed in Tables 1, 2, 3 are perhaps excessively optimistic.

3. Surveying the database

Our census lets us survey several interesting invariants associated to cubic fourfolds over $\mathbb{F}_2$. We now outline the main invariants studied in this section. Our dataset includes the $\mathbb{F}_2$-automorphism groups of every cubic fourfold over $\mathbb{F}_2$, which happens to coincide with $\mathrm{GL}_6(\mathbb{F}_2)$-stabilizer that we compute in the course of running the filtration method. Some highlights of this automorphism data are presented in §3.1. In §3.2, the orders of the automorphism groups are used to count $\mathbb{F}_2$-points on the *discriminant complement* of the moduli space of cubic hypersurfaces in $\mathbb{P}^5$; we relate this count to work of Poonen [44], Church–Ellenberg–Farb [12],

TABLE 1. List of feasible cases for $q = 2$. Too many orbits indicated by X; and Y, N indicate yes, no, respectively. The left symbol is for a naive orbit partition algorithm, and the right symbol is using the filtration method.

$n\backslash d$	2	3	4	5	6	7	8	9	$\cdots$	48	49
0	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	$\cdots$	Y|Y	X
1	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	X	$\cdots$	X	X
2	Y|Y	Y|Y	Y|Y	N|Y	X	X	X	X	$\cdots$	X	X
3	Y|Y	Y|Y	X	X	X	X	X	X	$\cdots$	X	X
4	Y|Y	N|Y	X	X	X	X	X	X	$\cdots$	X	X
5	Y|Y	N|Y	X	X	X	X	X	X	$\cdots$	X	X
6	Y|Y	X	X	X	X	X	X	X	$\cdots$	X	X
7	Y|Y	X	X	X	X	X	X	X	$\cdots$	X	X
8	N|Y	X	X	X	X	X	X	X	$\cdots$	X	X
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	N|Y	X	X	X	X	X	X	X	$\cdots$	X	X

TABLE 2. List of feasible cases for $q = 3$. Too many orbits indicated by X; and Y, N indicate yes, no, respectively. The left symbol is for basic union-find, and the right symbol is using the filtration method.

$n\backslash d$	2	3	4	5	6	7	8	9	$\cdots$	31	32
0	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	$\cdots$	Y|Y	X
1	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	N|Y	X	X	$\cdots$	X	X
2	Y|Y	Y|Y	N|Y	X	X	X	X	X	$\cdots$	X	X
3	Y|Y	N|Y	X	X	X	X	X	X	$\cdots$	X	X
4	Y|Y	N|N	X	X	X	X	X	X	$\cdots$	X	X
5	Y|Y	X	X	X	X	X	X	X	$\cdots$	X	X
6	N|N	X	X	X	X	X	X	X	$\cdots$	X	X

TABLE 3. List of feasible cases for $q = 5$. Too many orbits indicated by X; and Y, N indicate yes, no, respectively. The left symbol is for basic naive orbit partition algorithm, and the right symbol is using the filtration method.

$n\backslash d$	2	3	4	5	6	7	8	9	$\cdots$	22	23
0	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	Y|Y	$\cdots$	Y|Y	X
1	Y|Y	Y|Y	Y|Y	Y|Y	N|Y	X	X	X	$\cdots$	X	X
2	Y|Y	Y|Y	X	X	X	X	X	X	$\cdots$	X	X
3	Y|Y	N|N	X	X	X	X	X	X	$\cdots$	X	X
4	Y|Y	X	X	X	X	X	X	X	$\cdots$	X	X
5	N|N	X	X	X	X	X	X	X	$\cdots$	X	X

Vakil–Wood [49], and Howe [29]. Finally, in §3.3, we compute all $\mathbb{F}_2$-lines and planes on cubics in our database, verifying statistics predicted by Debarre–Laface–Roulleau [14] and giving a lower bound on the number of smooth *rational* cubic fourfolds over $\mathbb{F}_2$.

3.1. **Automorphisms of cubics.** The automorphism groups of cubic hypersurfaces over various fields have been well-studied. Over the complex numbers, the *symplectic* automorphism groups of smooth cubic fourfolds were recently fully classified by Laza and Zheng [36], and they additionally prove that the Fermat cubic has the largest automorphism group of any smooth cubic fourfold over $\mathbb{C}$ (see also

[43]). In positive characteristic, we do not know of any body of literature on the automorphism groups of cubic hypersurfaces of dimension > 2. The automorphism groups of cubic *surfaces* over algebraically closed fields of *arbitrary characteristic* were completely classified by Dolgachev and Duncan [15]. Our orbit-finding method yields a complete classification of the $\mathbb{F}_2$-automorphism groups of cubic surfaces, threefolds, and fourfolds. We report some specific results on cubic fourfolds here.

First, we compare the automorphism group of a hypersurface with its stabilizer subgroup.

Proposition 3.1. *Let k be a field and assume that $n \geq 3$ and $d \geq 3$. Let $f \in \mathrm{Sym}^d(k^{n+2})$ be nonzero and $X \subset \mathbb{P}_k^{n+1}$ be the associated projective hypersurface. Then the $\mathrm{PGL}_{n+2}(k)$-stabilizer of the line spanned by f is isomorphic to the group $\mathrm{Aut}_k(X)$ of k-automorphisms X.*

Moreover, if every element in $k^\times$ is a dth power (e.g., if $k = \mathbb{F}_q$ and $q-1$ is relatively prime to d) then the $\mathrm{GL}_{n+2}(k)$-stabilizer of f is isomorphic to the group $\mathrm{Aut}_k(X)$ of k-automorphisms X.

Proof. The first statement follows immediately from Proposition 2.1. As for the second statement, if $G_f \subset \mathrm{GL}_{n+2}$ denotes the stabilizer k-subgroup scheme of f, then we have a short exact sequence of k-group schemes

$$1 \to \mu_d \to G_f \to \mathrm{Aut}_k(X) \to 1,$$

where here, $\mathrm{Aut}_k(X)$ is considered as a constant group scheme. The associated exact sequence in flat cohomology starts

$$1 \to \mu_d(k) \to G_f(k) \to \mathrm{Aut}(X) \to H^1(k, \mu_d).$$

Under the hypotheses that every element in $k^\times$ is a dth power, we have that $\mu_d(k)$ and $H^1(k, \mu_d)$ are trivial. The statement then follows since $G_f(k)$ coincides with the $\mathrm{GL}_{n+2}(k)$-stabilizer subgroup of f. Indeed, the exact sequence in flat cohomology associated to the stabilizer subgroup group scheme starts

$$1 \to G_f(k) \to \mathrm{GL}_{n+2}(k) \to (\mathrm{GL}_{n+2}.f)(k) \to H^1(k, G_f)$$

and, since f is a k-rational point in the orbit, $G_f(k)$ is precisely the subset of elements of $\mathrm{GL}_{n+2}(k)$ acting trivially on f. □

Hence in the case of cubic fourfolds over $\mathbb{F}_2$, the stabilizer of a cubic form coincides with the $\mathbb{F}_2$-automorphism group of its associated hypersurface. To compute the $G = \mathrm{GL}_6(\mathbb{F}_2)$-stabilizer of the cubic form $f \in V$ that defines X, we use the same idea behind Algorithm 1.3 to successively reduce the order of the acting group: letting $\pi_1 \colon V \to V/W_1$ and $\pi_2 \colon V/W_1 \to V/W_2$ denote the natural projections, Lemma 1.1(3) tells us that

$$G_f = (G_{\pi_1(f)})_f = ((G_{\pi_2(f)})_{\pi_1(f)})_f.$$

Remark 3.2 (The isomorphism problem). A similar application of the filtration method gives an efficient algorithm for solving the cubic isomorphism problem over $\mathbb{F}_2$: our library includes an intrinsic `IsEquivalentCubics` which determines whether two cubic fourfolds are $\mathbb{F}_2$-isomorphic, and if they are, returns an explicit isomorphism between them. It runs in about 0.2 seconds per pair of cubics on a household laptop.

We now discuss some highlights emerging from our computation of the automorphism groups of cubic fourfolds over $\mathbb{F}_2$. It is a well-known result that the automorphism group of a generic hypersurface of dimension $n \geq 2$ and degree $d \geq 3$, over an algebraically closed field, is trivial (see [41]). Indeed, our survey shows that $\mathrm{Aut}_{\mathbb{F}_2}(X) = \{\mathrm{id}\}$ for most cubic fourfolds.

Computation 3.3. *Among the* $3\,718\,649$ *isomorphism classes of cubic fourfolds over* $\mathbb{F}_2$*, there are* $3\,455\,271$*, or about* 92.9%*, with trivial stabilizer. Among the* $1\,069\,562$ *isomorphism classes of smooth cubic fourfolds over* $\mathbb{F}_2$*, there are* $1\,029\,478$*, or about* 96.3%*, with trivial stabilizer.*

We summarize our computation of all of the nontrivial automorphism groups of cubic fourfolds in the next theorem.

Theorem 3.4. *If* X *is cubic fourfold over* $\mathbb{F}_2$*, then the order of* $\mathrm{Aut}_{\mathbb{F}_2}(X)$ *is one of the following 87 possibilities:*

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 16, 18, 20, 21, 24, 30, 32, 36, 42, 48, 60, 63, 64, 72, 84, 96, 108, 120, 126, 128, 144, 160, 168, 192, 256, 288, 320, 384, 512, 576, 648, 672, 720, 768, 882, 1024, 1152, 1344, 1440, 1536, 1920, 2016, 2048, 2160, 2304, 3072, 3840, 4032, 4608, 6144, 7680, 9216, 10752, 11520, 12288, 18432, 23040, 24576, 27648, 32256, 36864, 73728, 86016, 172032, 258048, 344064, 516096, 1105920, 1451520, 1806336, 5160960, 10321920, 15482880, 30965760, 319979520.

If X *is a smooth cubic fourfold over* $\mathbb{F}_2$*, then the order of* $\mathrm{Aut}_{\mathbb{F}_2}(X)$ *is one of the following 40 possibilities:*

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 16, 18, 24, 30, 32, 36, 48, 64, 72, 96, 108, 120, 128, 144, 160, 192, 288, 384, 512, 576, 648, 1440, 1536, 2160, 4608, 10752, 23040, 1451520.

Remark 3.5 (An extremal cubic fourfold). The smooth cubic fourfold

$$X_1: \; x_0x_3^2 + x_1x_4^2 + x_2x_5^2 + x_0^2x_3 + x_1^2x_4 + x_2^2x_5 = 0 \tag{5}$$

is the unique cubic fourfold over $\mathbb{F}_2$ with $|\mathrm{Aut}(X_1)(\mathbb{F}_2)| = 1\,451\,520$; in fact, our stabilizer computations yield that the $\mathbb{F}_2$-automorphisms form a group isomorphic to the symplectic group $\mathrm{Sp}(6, \mathbb{F}_2)$, and this is the largest $\mathbb{F}_2$-automorphism group of all smooth $\mathbb{F}_2$-cubic fourfolds. This cubic fourfold was also studied in [16, 31].

The appearance of the symplectic group admits a simple explanation. We consider the $\mathbb{F}_4/\mathbb{F}_2$-Hermitian form defined by

$$H(x, y) := x_0y_5^2 + x_1y_4^2 + x_2y_3^2 + x_3y_4^2 + x_4y_1^2 + x_5y_0^2,$$

where $x = (x_0, \ldots, x_5)$ and $y = (y_0, \ldots, y_5)$ are in $\mathbb{F}_4^6$. The map from Hermitian forms to cubic forms defined by $H(x, y) \mapsto H(x, x)$ is injective, so in particular, $g \in \mathrm{GL}_6(\mathbb{F}_4)$ fixes $H(x, y)$ if and only if it fixes $H(x, x)$. We observe that the group of $\mathbb{F}_2$-rational points of the unitary group of H is isomorphic to $\mathrm{Sp}(6, \mathbb{F}_2)$.

Remark 3.6 (The Fermat cubic fourfold). There is also a unique smooth cubic fourfold X_2 with $|\mathrm{Aut}(X_2)(\mathbb{F}_2)| = 23040$, the Fermat cubic fourfold

$$X_2: \; x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0.$$

Incidentally, there are also three singular cubic fourfolds with an automorphism group of this order, two of which have $\mathbb{F}_2$-automorphism group isomorphic to $\mathrm{Aut}_{\mathbb{F}_2}(X_2)$. One easily sees that the cubic X_1 is an $\mathbb{F}_2$-form of the Fermat cubic X_2, split by any $K/\mathbb{F}_2$ containing a primitive third root of unity; in particular, $X_1 \times_{\mathbb{F}_2} \mathbb{F}_4 \cong X_2 \times_{\mathbb{F}_2} \mathbb{F}_4$.

3.2. **Point counting on moduli spaces.** Let U_d be the the discriminant complement in the Hilbert scheme of degree d hypersurfaces over $\mathbb{F}_q$, i.e., U_d is the open subscheme of $\mathbb{P}_{\mathbb{F}_q}^{\binom{d+n}{d}-1}$ parametrizing smooth degree d hypersurfaces in $\mathbb{P}^n_{\mathbb{F}_q}$. A relevant question is: what is the probability that a randomly chosen hypersurface is smooth? In [44], Poonen gave an answer asymptotically in d, proving that

$$\lim_{d\to\infty} \frac{|U_d(\mathbb{F}_q)|}{|\mathbb{P}^{\binom{d+n}{d}-1}(\mathbb{F}_q)|} = \frac{1}{Z_{\mathbb{P}^n}(n+1)} = \prod_{1\le k\le n}\left(1-\frac{1}{q^k}\right).$$

Poonen's result on point counting on discriminant complements has been related to the phenomena of stabilization in the Grothendieck ring in work of Vakil and Wood [49], and has also been studied from the perspective of homological and representation stability by Church, Ellenberg, and Farb [12] and by Howe [29].

In the case of hypersurfaces in $\mathbb{P}^5$ over $\mathbb{F}_2$, Poonen's result says that the probability as $d \to \infty$ that a randomly chosen hypersurface of degree d is smooth should be

$$\prod_{1\le k\le 5}\left(1-\frac{1}{2^k}\right) = 0.298004150390625. \tag{6}$$

Our computations let us compute $|U_3(\mathbb{F}_2)|$ exactly, and therefore the probability that a randomly chosen *cubic* hypersurface in $\mathbb{P}^5_{\mathbb{F}_2}$ is smooth: summing the sizes of the orbits $[\mathrm{PGL}_6(\mathbb{F}_2) : \mathrm{Aut}_{\mathbb{F}_2}(X)]$ gives the $\mathbb{F}_2$-point count on the discriminant complement U_3.

Theorem 3.7. *The cardinality of the set of smooth cubic fourfolds over $\mathbb{F}_2$ (not considered up to isomorphism) is*

$$|U_3(\mathbb{F}_2)| = 21\,138\,040\,038\,850\,560,$$

and thus the probability that a random cubic fourfold over $\mathbb{F}_2$ is smooth is exactly

$$\frac{|U_3(\mathbb{F}_2)|}{|\mathbb{P}^{55}(\mathbb{F}_2)|} = 0.29334923433225412736646831035614013671875.$$

We compare this count to some other counts of the proportion of smooth small degree hypersurfaces over $\mathbb{F}_2$, see Table 4. The proportion of cubic fourfolds which are smooth is closer to the Poonen limit than any proportion we could compute.

We can also give some point counts on a series of related moduli spaces:

- $\mathcal{C}^{sm}$, the coarse moduli space of smooth cubic fourfolds,
- $\mathscr{C}$, the moduli stack of cubic fourfolds, and
- $\mathscr{C}^{sm}$, the moduli stack of smooth cubic fourfolds.

Using our computations of the automorphism groups discussed above, we determine the stacky point counts:

$$|\mathscr{C}(\mathbb{F}_2)| = \sum_{X\in C_3} \frac{1}{|\mathrm{Aut}_{\mathbb{F}_2}(X)|} = \frac{4803839602528529}{1343913984} \approx 3\,574\,514.18746,$$

TABLE 4. List of proportion of hypersurfaces of degree d in $\mathbb{P}^{n+1}$ that are smooth, over $\mathbb{F}_2$. The last column is the asymptotic proportion as $d \to \infty$ from Poonen's theorem.

$n \backslash d$	1	2	3	4	5	$\prod_{1 \le k \le n+1} \left(1 - \frac{1}{2^k}\right)$
1	1	$\frac{4}{9} \approx 0.444$	$\frac{112}{341} \approx 0.328$	$\frac{1560}{4681} \approx 0.333$	$\frac{98304}{299593} \approx 0.328$	$\frac{3}{8} = 0.375$
2	1	$\frac{448}{1023} \approx 0.438$	$\frac{21504}{69905} \approx 0.308$	$\frac{10590854400}{34359738367} \approx 0.308$		$\frac{21}{64} \approx 0.328$
3	1	$\frac{64}{151} \approx 0.424$	$\frac{330301440}{1108378657} \approx 0.298$			$\frac{315}{1024} \approx 0.308$
4	1	$\frac{126976}{299593} \approx 0.424$	$\frac{1409202669256704}{4803839602528529} \approx 0.293$			$\frac{9765}{32768} \approx 0.298$

where C_3 is a complete set of $\mathbb{F}_2$-isomorphism classes of cubic fourfolds X over $\mathbb{F}_2$, and

$$|\mathscr{C}^{\mathrm{sm}}(\mathbb{F}_2)| = \sum_{X \in C_3^{\mathrm{sm}}} \frac{1}{|\mathrm{Aut}_{\mathbb{F}_2}(X)|} = 1\,048\,581,$$

where C_3^{sm} is the subset of smooth isomorphism classes.

Furthermore, there is an equality of point counts

$$|\mathcal{C}^{sm}(\mathbb{F}_2)| = |\mathscr{C}^{\mathrm{sm}}(\mathbb{F}_2)|,$$

because $\mathscr{C}^{\mathrm{sm}}$ is a DM stack, and therefore its stacky point count is the same as the point count of its coarse space, see [6, Propositon 1.3]. Our methods still leave open the problem of calculating $|\mathcal{C}(\mathbb{F}_2)|$, where $\mathcal{C}$ is the course moduli space of all cubic fourfolds.

3.3. **Lines on cubic fourfolds.** We compute the set of $\mathbb{F}_2$-lines on every cubic fourfold. We find that there exist exactly 65 cubic fourfolds which contain exactly one $\mathbb{F}_2$-line, only 29 of which are smooth (the first example of such a cubic was given in [14]). Our exhaustive computations confirm that every cubic fourfold X contains an odd number of $\mathbb{F}_2$-lines (so in particular they all contain at least one such line). In fact, this was already proved by Debarre–Laface–Roulleau.

Lemma 3.8 ([14])**.** *Every cubic fourfold over $\mathbb{F}_2$ contains a line defined over $\mathbb{F}_2$.*

In [14], this result is derived using a formula of Galkin and Shinder ([14, Equation 8], [22]): there is a relation between point counts on X and its Fano variety of lines $F_1(X)$, which in the case of cubic fourfolds over $\mathbb{F}_2$ yields

$$|F_1(X)(\mathbb{F}_2)| = \frac{|X(\mathbb{F}_2)|^2 - 2(1+2^4)|X(\mathbb{F}_2)| + |X(\mathbb{F}_4)|}{2 \cdot 2^2} \; + \; 4\,|\mathrm{Sing}(X)(\mathbb{F}_2)|. \tag{7}$$

In particular, since $|X(\mathbb{F}_2)| \geq 1$ (by Chevalley–Warning), one has $|F_1(X)(\mathbb{F}_2)| \geq 1$.

If one is only interested in the *number* of lines on a cubic fourfold then the above formula suffices provided one computes point counts on X (as we do in §4), but we still computed the full set of $\mathbb{F}_2$-lines on each cubic with other applications in mind—for instance, if one is interested in searching for various *families* of lines on cubics, like planes and scrolls, it is useful to know the full set of lines.

We plot a histogram, see Figure 1, of the count (weighted by stabilizer) of the number of isomorphism classes of cubics containing a given number of lines. We also plot the same histogram for just the smooth cubics. The histograms match the prediction from [14].

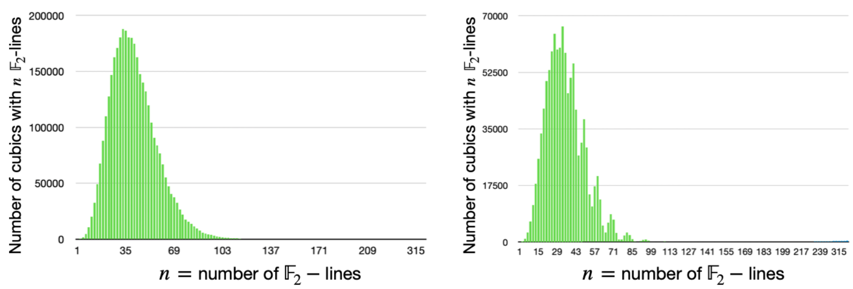

FIGURE 1. Stacky counts of the number of cubics containing n lines. A plot restricting to smooth cubics is given on the right.

Remark 3.9. The maximal number of lines on a smooth cubic fourfold over $\mathbb{F}_2$ is 315. The extremal cubic fourfold X_1 in Remark 3.5 is the unique smooth cubic fourfold over $\mathbb{F}_2$ with 315 lines.

The most likely number of lines on a cubic fourfold over $\mathbb{F}_2$, and on a smooth cubic fourfold over $\mathbb{F}_2$, is 33 (i.e., both histograms in Figure 1 have a peak at 33). The expected number of lines on a smooth cubic fourfold over $\mathbb{F}_2$, with respect to the uniform distribution on $U_3(\mathbb{F}_2)$, is ≈ 10.2667.

There are "bumps" on these two histograms, which become significantly more pronounced in the figure on the right. We do not know how to entirely explain these bumps, but we remark that the sparseness of the smooth cubic fourfolds in the database partially explains why the gaps are larger on the right.

3.4. **Planes on cubic fourfolds.** We also compute the complete set of $\mathbb{F}_2$-planes on every cubic fourfold over $\mathbb{F}_2$. In contrast to the case of lines, not every cubic has a plane; indeed, cubic fourfolds containing plane live on the Noether–Lefschetz divisor $\mathcal{C}_8 \subset \mathcal{C}$. There are $2\,116\,029$ cubic fourfolds, or 56.90% up to isomorphism, containing at least one $\mathbb{F}_2$-plane, of which $702,153$ are smooth, or 65.65% of the smooth cubic fourfolds up to isomorphism. Figure 2 shows the histograms recording how many cubic fourfolds (respectively, smooth cubic fourfolds) contain a fixed number of $\mathbb{F}_2$-planes.

As mentioned in the introduction, the rationality problem for cubic fourfolds has been a primary motivation for their study over the last 80 years (see [19], [5], and the survey [26]). There is a well-known rationality construction for any cubic X containing a pair of disjoint $\mathbb{F}_2$-planes P_1 and P_2: X is birational to $P_1 \times P_2$. Counting the cubics in our database with pairs of disjoint $\mathbb{F}_2$ planes gives a lower bound on the number of rational cubic fourfolds over $\mathbb{F}_2$:

Computation 3.10. *There are* $429\,744$ *isomorphism classes of cubic fourfolds in* $\mathcal{C}(\mathbb{F}_2)$ *containing two disjoint* $\mathbb{F}_2$*-planes, and* $36\,572$ *of these cubics are smooth.*

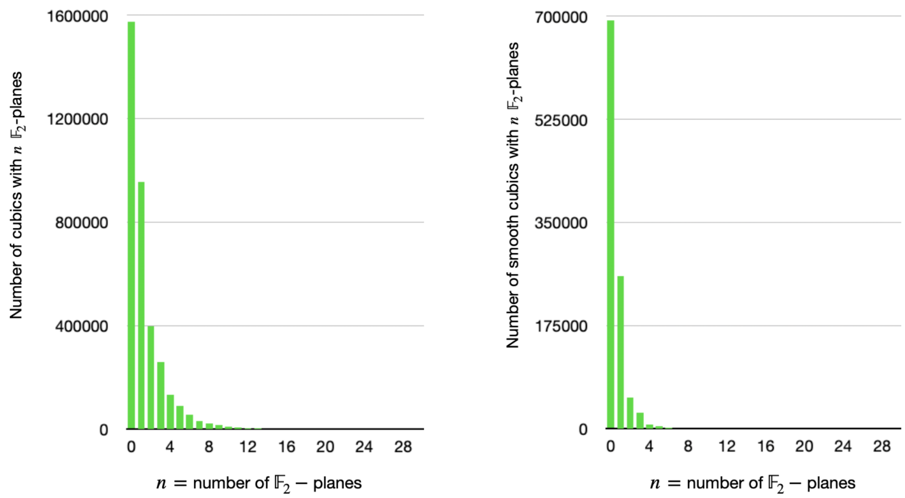

FIGURE 2. Stacky counts of the number of cubics containing n planes. A plot restricting to smooth cubics is given on the right.

4. ZETA FUNCTIONS OF CUBIC FOURFOLDS

In this section we explain how we compute the zeta functions of all smooth cubic fourfolds over $\mathbb{F}_2$. The reader interested in the results of the computation is advised to skip to §4.3.

4.1. **Computational methods.** Let $q = 2$, and let $F\colon X_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$ be the relative Frobenius endomorphism. By the Weil conjectures, the zeta function for a smooth cubic fourfold $X/\mathbb{F}_q$ is given by

$$Z_X(q^{-s}) = \frac{1}{(1-q^{-s})(1-q^{1-s})(1-q^{2-s})Q_X(q^{-s})(1-q^{3-s})(1-q^{4-s})} = \frac{Z_{\mathbb{P}^4}(q^{-s})}{Q_X(q^{-s})},$$

where the interesting factor $Q_X(t)$ is given by, for an odd prime ℓ,

$$Q_X(t) = \det\bigl(\mathrm{Id} - tF^* \mid H^4_{\text{ét,prim}}(X_{\overline{\mathbb{F}}_2}, \mathbb{Q}_\ell)\bigr).$$

Let us now describe how we efficiently compute the zeta functions of cubic fourfolds. Our code computes the Weil polynomials

$$P_X(t) = \det\bigl(F^* - t\mathrm{Id} \mid H^4_{\text{ét,prim}}(X_{\overline{\mathbb{F}}_2}, \mathbb{Q}_\ell(2))\bigr)$$

for each isomorphism class of smooth cubic fourfold $X/\mathbb{F}_2$ in our database; each $P_X(t)$ is a monic, degree 22 polynomial with coefficients in $\frac{1}{2}\mathbb{Z}$ whose roots lie on the unit circle. The Weil polynomial $P_X(t)$ is related to $Q_X(t)$ via $P_X(t) = \pm Q_X(t/4)$ where the sign is the sign occurring in the functional equation, see (8). Using a slight adaptation of the algorithm of Addington–Auel in [1] (described below), we can efficiently compute the point counts of smooth (and even mildly singular) cubic fourfolds over $\mathbb{F}_2$. Then for each smooth cubic X, we compute the first 11 nonleading coefficients of $P(t)$ using the points counts $|X(\mathbb{F}_{2^k})|$ for $1 \le k \le 11$.

The 11 remaining coefficients are computed by leveraging the functional equation

$$P_X(t) = \varepsilon t^{22} P_X(t^{-1}), \tag{8}$$

where $\varepsilon \in \{\pm 1\}$ is the sign of the functional equation, to fully determine $P_X(t)$. In order to determine ε, we use work of T. Saito [45] to relate it to the divided discriminant $\mathrm{disc}_d(f)$ of an integral homogeneous cubic lift $f \in \mathbb{Z}[x_0, \ldots, x_5]$, see [45, Definition 2.2] for the definition of $\mathrm{disc}_d(f)$.

Theorem 4.1 ([45, §4])**.** *Let X be a smooth cubic fourfold over $\mathbb{F}_2$, and let $f \in \mathbb{Z}[x_0, \ldots, x_5]$ be a lift of a defining equation for X. Then the sign of the functional equation in the zeta function of X is $(-1)^{(\mathrm{disc}_d(f)+1)/4}$.*

Proof. We first remark that, by (8), the determinant of the action of Frobenius on $H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_2}, \mathbb{Q}_\ell(2))$ is the sign of the function equation of the zeta function of X. Thus the determinant Galois representation

$$\det H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_2}, \mathbb{Q}_\ell(2)) : \mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2) \to \{\pm 1\}$$

is a quadratic character, which is nontrivial precisely when $\varepsilon = -1$. By [45, Theorem 4.2 and Corollary 4.3], this is equivalent to $B(f)A(f)^{-2} \equiv 1$ in $\mathbb{F}_2$, where $A(f), B(f) \in \mathbb{Z}$ satisfy $-\mathrm{disc}_d(f) = A(f)^2 + 4B(f)$; we note that in our case, where $n = 4$ and $d = 3$, we have that $\varepsilon(n, d) = -1$ in the notation of [45, (3.5.1)]. Modulo 4, we can take $A(f) = 1$ and $B(f) \in \{0, 1\}$, hence we have that the sign of the function equation is $(-1)^{B(f)} = (-1)^{(\mathrm{disc}_d(f)+1)/4}$. □

Applying Saito's criterion to the smooth cubics in our database, one finds that nearly half of all smooth cubic fourfolds take the positive sign in their functional equation.

Computation 4.2. *Among the* 1 069 562 *isomorphism classes of smooth cubic fourfolds over $\mathbb{F}_2$, there are exactly* 531 334*, or about* 49.8%*, for which the sign of the functional equation is* $+1$.

4.2. **Addington–Auel point counting algorithm.** For the sake of documentation, we describe our adaptations to the point counting algorithm of Addington and Auel [1, §3], the idea of which itself goes back (for counting points on cubic threefolds) to Bombieri–Swinnerton-Dyer [7] and was used by Debarre–Laface–Roulleau [14, §4.3]. Our improvements are as follows. First, we remove the hypothesis that X is smooth. Second, we no longer require that the cubic fourfold X contains an $\mathbb{F}_2$-line not also contained in a plane $P \subset X \times_{\mathbb{F}_2} \overline{\mathbb{F}}_2$. (Nevertheless, there are only 55 smooth cubic fourfolds over $\mathbb{F}_2$ where this fails.)

The key step in the algorithm is to transform X into a conic fibration. We denote $\mathbb{P}^5 = \mathrm{Proj}\,\mathbb{F}_2[y_0, \ldots, y_5]$ and $X \subset \mathbb{P}^5$ an arbitrary cubic fourfold. We let $\ell \subset X$ be a line (such a line must exist by Lemma 3.8) and change coordinates so that $\ell = \{y_0 = y_1 = y_2 = y_3 = 0\}$. Then the defining equation of X is

$$\begin{aligned} &A(y_0, \ldots, y_3)y_4^2 + B(y_0, \ldots, y_3)y_4 y_5 + C(y_0, \ldots, y_3)y_5^2 \\ &+D(y_0, \ldots, y_3)y_4 + E(y_0, \ldots, y_3)y_5 \quad + F(y_0, \ldots, y_3) \quad = 0, \end{aligned} \tag{9}$$

where $A, \ldots, F$ are homogeneous polynomials of degrees $1, 1, 1, 2, 2, 3$, respectively.

Consider the projection $\phi : X \dashrightarrow \mathbb{P}^3 = \mathrm{Proj}\,\mathbb{F}_2[y_0, \ldots, y_3]$ away from ℓ. The points of the base $\mathbb{P}^3$ correspond to the projective 2-planes P in $\mathbb{P}^5$ containing ℓ. For any plane $P \supset \ell$ not contained in X, the scheme theoretic intersection of P and X is $\ell \cup C$, where C is a plane conic, and thus the fiber $\phi^{-1}([P])$ is C. Hence the resolution $\phi\colon \widetilde{X} \to \mathbb{P}^3$, where $\pi : \widetilde{X} \to X$ is the blow up of X along ℓ, is a conic fibration.

Our algorithm will also require us to compute the point counts for the exceptional locus of π. Since π is the blow up along a line, $\pi^{-1}(x)$ is a linear space for every point $x \in X$. Specifically,

$$\pi^{-1}(x) \cong \begin{cases} \mathbb{P}^0 & \text{if } x \notin \ell \\ \mathbb{P}^2 & \text{if } x \in \ell, x \notin X^{\text{sing}} \\ \mathbb{P}^3 & \text{otherwise} \end{cases}.$$

More precisely, the blow up $\widetilde{\mathbb{P}}^5$ of $\mathbb{P}^5$ along ℓ has exceptional divisor $\mathbb{P}_\ell(N_{\ell/\mathbb{P}^5}) = \ell \times \mathbb{P}^3 \to \ell$. Thus the exceptional divisor of $\widetilde{X}$, being the strict transform of X in $\widetilde{\mathbb{P}}^5$, is a divisor in $\ell \times \mathbb{P}^3$. When $x \in \ell$ is a smooth point of X then $\pi^{-1}(x) \subset \{x\} \times \mathbb{P}^3$ is the $\mathbb{P}^2$ of points in $\mathbb{P}^3$ corresponding to 2-planes through ℓ tangent to X at x. However, when $x \in \ell$ is a singular point of X, the fiber $\pi^{-1}(x)$ picks out all the planes through ℓ, i.e., the fiber is the entire $\mathbb{P}^3$.

Note that $x \in \ell$ is a singular point of X if and only if it a basepoint of the 3-dimensional family of conics defined in (9). Thus, depending on A, B, C, we have one of the following cases for the basepoints that lie in on ℓ:

- 0 basepoints
- 1 basepoint
- 2 basepoints (both defined over $\mathbb{F}_2$)
- 2 basepoints (neither defined over $\mathbb{F}_2$)
- A line of basepoints ($A = B = C = 0$).

It is thus enough to know these basepoints on the line ℓ in order to compute the point counts for the exceptional divisor, and most of the runtime will be devoted to point counting on the conic fibration.

Algorithm 4.3. `CountPoints`(X, q) (Adapted from [1, §3])
Input:

- A cubic fourfold X
- $q = 2^r$

Steps:

(1) Choose an $\mathbb{F}_2$-line $\ell \subset X$ (guaranteed by Lemma 3.8).
(2) The projection away from ℓ yields a morphism $\phi\colon \mathrm{Bl}_\ell(X) \to \mathbb{P}^3$ whose generic fiber is a conic, as in equation (9).
(3) If $A = B = C = D = E = 0$, then ϕ gives X the structure of a cone over a cubic surface $Y = \{F = 0\} \subset \mathbb{P}^3$. In this case

    return $1 + q + |Y(\mathbb{F}_q)| \cdot q^2$.

(4) Compute $|\mathrm{Bl}_\ell(X)(\mathbb{F}_q)|$ by counting $\mathbb{F}_q$-points in each fiber of ϕ: let

$$\Delta = \{AE^2 + B^2F + CD^2 - BDE = 0\} \subseteq \mathbb{P}^3$$

    be the discriminant subscheme parametrizing the fibers of ϕ which are either planes or singular conics. For each $y \in \Delta(\mathbb{F}_q)$, the point count $|(\phi^{-1}(y))(\mathbb{F}_q)|$ is determined as follows:

- if $y \in \Delta$ and $y \notin \{A = \cdots = F = 0\}$, then $\phi^{-1}(y)$ is a singular plane conic over $\mathbb{F}_q$. It has 0, $q+1$, or $2q+1$ $\mathbb{F}_q$-points, which can be determined by its rank and Arf invariant.

- if $y \in \{A = \cdots = F = 0\}$, then $\phi^{-1}(y)$ is a 2-plane over $\mathbb{F}_q$ with $1+q+q^2$ $\mathbb{F}_q$-points.

Then

$$|\mathrm{Bl}_\ell(X)(\mathbb{F}_q)| = \sum_{x \in \Delta(\mathbb{F}_q)} |(\phi^{-1}(x))(\mathbb{F}_q)| + (q+1)(1+q+q^2+q^3 - |\Delta(\mathbb{F}_q)|).$$

(5) Determine the exceptional divisor T of π.
- If $x \in \ell$ is not a singular point of X, $|\pi^{-1}(x)(\mathbb{F}_q)| = 1+q+q^2$.
- If $x \in \ell$ is a singular point, $|\pi^{-1}(x)(\mathbb{F}_q)| = 1+q+q^2+q^3$.

In the case that $\ell \subset X^{\mathrm{sm}}$, we have $|T(\mathbb{F}_q)| = (1+q)(1+q+q^2)$. Thus, we may compute

$$|\mathrm{Bl}_\ell(X)(\mathbb{F}_q)| = |X(\mathbb{F}_q)| + |T(\mathbb{F}_q)| - |\mathbb{P}^1(\mathbb{F}_q)|$$

return $|X(\mathbb{F}_q)|$.

As in the original algorithm in [1], we gain significant speedup in the enumeration of $\Delta(\mathbb{F}_q)$ by taking the projection $\Delta\backslash\{p\} \to \mathbb{P}^2$ away from a singular point $p \in \Delta^{\mathrm{sing}}(\mathbb{F}_2)$ and enumerating over $\mathbb{P}^2(\mathbb{F}_q)$. In the process of running the point-counting algorithm, we discovered that we can always find such a singular point, which answers a question of [1] (see footnote 4 of *loc. cit.*). In fact, we can give a direct proof of this over any field.

Proposition 4.4. *Let X be a smooth cubic fourfold containing a line over a field k. Then X contains a (possibly different) line such that the discriminant of the associated conic fibration has a singular k-point.*

Proof. Let $\ell \subset X$ be a line defined over k. We remark that any line defined over any field contains at least three rational points, and we choose three such points. The intersection of the projectivized tangent spaces (which are projective 4-planes over k since X is smooth) at these three points contains a projective 2-plane $P \subset \mathbb{P}^5$ over k. The plane P is tangent to X along ℓ. Thus the scheme theoretic intersection of P and X is plane cubic that contains ℓ with multiplicity (at least) 2, hence contains another line ℓ' (which may be equal to ℓ). By definition, the fiber of the conic fibration associated to ℓ', over the point of $\mathbb{P}^3$ corresponding to P, is a double line, hence the discriminant has a k-rational singular point. □

In fact, in Proposition 4.4, we did not need to assume that X is smooth, but only that X contains a line that meets the smooth locus in at least 3 rational points.

4.3. **Census of the zeta functions.** Our computations of all the zeta functions of smooth cubic fourfolds over $\mathbb{F}_2$ yield the following result.

Computation 4.5. *There are* 86 472 *distinct zeta functions realized among the* 1 069 562 *isomorphism classes of smooth cubic fourfolds over* $\mathbb{F}_2$.

In doing the computation above, we also verified a conjecture of Elsenhans and Jahnel [18, Theorem 1.9] in the case of cubic fourfolds over $\mathbb{F}_2$.

Theorem 4.6. *Let X be a smooth cubic fourfold defined over $\mathbb{F}_2$, and $P_X(t)$ its primitive Weil polynomial. Then $2P_X(-1)$ is an integer square.*

4.4. **The $K3$-part of the Weil polynomial of a cubic fourfold.** If $(1-t)$ divides the Weil polynomial $P_X(t)$ of some cubic fourfold X, then $P_X(t)/(1-t)$ is a degree 21 Weil polynomial, which we call the $K3$-part of the Weil polynomial of the cubic fourfold.

Computation 4.7. *We compared the Weil polynomials $P_X(t)/(1-t)$ against the list generated by Kedlaya–Sutherland* [33, Computation 3(c)] *of Weil polynomials of degree 21 which are of "K3-type", finding that there are* 71 476 *$K3$-type Weil polynomials which are the $K3$-part of the Weil polynomial of some cubic fourfold over $\mathbb{F}_2$.*

As expected, Computation 4.7 shows that cubic fourfolds produce $K3$-type Weil polynomials that are not realized by any quartic $K3$ over $\mathbb{F}_2$; indeed, there are only 52 755 degree 21 Weil polynomials arising from these quartic surfaces ([33, Computation 4(c)]). This fits in to the larger Honda–Tate program of giving geometric realizations for the Weil polynomials in [33, Computation 3(c)], see also [3].

These $K3$-type Weil polynomials arising form cubic fourfolds are sometimes (but not always) explained by the phenomenon of *associated $K3$ surfaces* mentioned in the introduction. For instance, whenever there is a geometric construction of an associated (twisted) $K3$ surface defined over the ground field k, the $K3$-part of the Weil polynomial of the cubic fourfold arises from an honest $K3$ surface over k. This is part of a more general phenomenon: whenever there is a k-linear Fourier–Mukai equivalence between the $K3$ category of X (as defined in [35], see [30]) and the derived category of (twisted) coherent sheaves on a $K3$ surface, Fu and Vial [20] show that the transcendental zeta functions of S and X agree (in fact, they have isomorphic rational Chow motives). We refer the interested reader to the excellent survey of Hassett [26] for more on associated $K3$ surfaces of cubic fourfolds.

4.5. **Newton polygons.** Having tabulated the zeta functions of the smooth cubic fourfolds over $\mathbb{F}_2$, we can determine their Newton polygons. The Newton polygon of a cubic fourfold over a finite field is determined by its height h, which can be any integer $1 \leq h \leq 10$, or $h = \infty$. We find smooth cubic fourfolds over $\mathbb{F}_2$ of every possible height, see Table 5 for the complete statistics.

Theorem 4.8. *Each Newton stratum in the moduli space of smooth cubic fourfolds contains $\mathbb{F}_2$-points.*

A cubic fourfold is called ordinary if $h = 1$ and supersingular if $h = \infty$.

Computation 4.9. *There are* 8688 *supersingular cubic fourfolds and* 533, 262 *ordinary cubic fourfolds up to isomorphism over $\mathbb{F}_2$.*

TABLE 5. Heights of isomorphism classes of cubic fourfolds over $\mathbb{F}_2$

h	1	2	3	4	5	6	7	8	9	10	∞
#	533262	267355	131922	66974	31806	16041	6901	4575	1301	737	8688

4.6. **Codimension 2 algebraic cycles on cubic fourfolds.** The Tate conjecture for $K3$ surfaces over finite fields of characteristic 2 has been proved by Ito–Ito–Koshikawa [32] and Kim–Madapusi Pera [34], [40]. We now explain how methods used to prove these results, as well as results for Gushel–Mukai varieties by Fu and Moonen [21], lead to a proof of the Tate conjecture for codimension 2 cycles on cubic fourfolds over $\mathbb{F}_2$. Since this result is surely known to the experts, we only provide a sketch of the proof to fill an existing gap in the literature.

Theorem 4.10. *Let X be a smooth cubic fourfold over a finite field k of characteristic 2. Then the cycle class map induces an isomorphism*

$$\mathrm{CH}^2(X) \otimes \mathbb{Q}_\ell \to H^4_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(2))^{\mathrm{Gal}(\overline{k}/k)}.$$

Proof. We use Madapusi Pera's approach to proving the Tate conjecture for codimension 2 cycles on smooth cubic fourfolds outlined in [39, §5.14], together with the integral 2-adic models of Shimura varieties constructed in [34], and the revised approach in [40], as adopted in [21].

Let $\mathscr{C}^{\mathrm{sm}}$ be the stack of smooth cubic fourfolds over $\mathbb{Z}_{(2)}$. Let $L \subset L'$ be the abstract lattice of the primitive part inside the H^4 of a cubic fourfold. Then L is even with signature $(20, 2)$ and discriminant 3 and L' is odd unimodular of signature $(21, 2)$. As in [39, §5.14], let $\widetilde{\mathscr{C}}^{\mathrm{sm}} \to \mathscr{C}^{\mathrm{sm}}$ be the double cover parameterizing cubic fourfolds together with a choice of isomorphism $\det(L) \otimes \mathbb{Z}_\ell \to \det(\langle h^2 \rangle^\perp)$, where $h^2 \in H^4_{\text{ét}}(X_{k^s}, \mathbb{Z}_\ell(2))$ is the cycle class of the square of the hyperplane section. Then by the arguments of [34, Proposition 4.15], the classical Kuga–Satake map extends to a morphism $\widetilde{\mathscr{C}}^{\mathrm{sm}} \to \mathcal{S}(L)$, where $\mathcal{S}(L)$ is a $\mathbb{Z}_{\mathfrak{p}}$-model of the orthogonal Shimura variety $\mathrm{Sh}(L)$ attached to L, see [34, Theorem 3.10]. Here, one needs to take a prime $\mathfrak{p}$ lying above 2 in an extension $E/\mathbb{Q}$ (of degree at most 2) that trivializes the quadratic character induced by the determinant-preserving Galois action on the primitive cohomology $\langle h^2 \rangle^\perp$, see [21, Remark 6.25 and §7.1]. This subtlety, which arises since the primitive cohomology has even rank, hence its special orthogonal group contains $\pm\mathrm{id}$, is not directly addressed in [39, §5.14]. However, to prove the Tate conjecture for codimension 2 cycles on X, we are free to take a finite extension of k, cf. [48, § 2, p. 580], namely, the residue field of $E_{\mathfrak{p}}$.

Following the strategy in [39], the main step is to show that the map $\widetilde{\mathscr{C}}^{\mathrm{sm}} \to \mathcal{S}(L)$ is étale. We remark that the de Rham realization of the universal lattice $\boldsymbol{L}_{\mathrm{dR}} \subset \boldsymbol{L}'_{\mathrm{dR}}$ is a vector subbundle since any polarization (having self-intersection 3) is primitive. Similarly, we appeal to [40, Lemma 1.10] to show that the map induced on de Rham realizations extends to an isometry to filtered vector bundles

$$\alpha_{\mathrm{dR}} : \boldsymbol{L}_{\mathrm{dR}}(-2) \to \boldsymbol{H}^4_{\mathrm{prim,dR}}$$

over $\widetilde{\mathscr{C}}^{\mathrm{sm}}$. The rest of the proof proceeds as in [39, §5.14], since $\widetilde{\mathscr{C}}^{\mathrm{sm}}$ is a smooth Artin stack, as proved by Levin [37, §3] (cf. [39, Proof of Theorem 5.15]), and the deformation theory of $\widetilde{\mathscr{C}}^{\mathrm{sm}}$ is controlled by the degree 4 part of the Hodge filtration on $\boldsymbol{H}^4_{\mathrm{prim,dR}}$. □

For a Weil polynomial $P(t)$ of a cubic fourfold X, we write

$$P(t) = P_{\mathrm{cyc}}(t) P_{\mathrm{non-cyc}}(t)$$

where $P_{\mathrm{cyc}}(t)$ is the product of all cyclotomic factors of $P(t)$. If $(t-1)^m$ exactly divides $P_{\mathrm{cyc}}(t)$ and $\deg P_{\mathrm{cyc}}(t) = n$, then we have, as a direct consequence of the

Tate conjecture, that

$$\operatorname{rk}\operatorname{CH}^2(X) = m \quad \text{and} \quad \operatorname{rk}\operatorname{CH}^2(\overline{X}) = n.$$

Thus, we can report the following statistics for the algebraic and geometric ranks for smooth cubic fourfolds over $\mathbb{F}_2$, see Tables 6 and 7.

TABLE 6. Rank of the group of algebraic cycles $\operatorname{CH}^2(X)$

$\operatorname{rk}\operatorname{CH}^2(X)$	1	2	3	4	5	6	7	8
how many	232218	426619	273007	106035	25521	5377	581	178

$\operatorname{rk}\operatorname{CH}^2(X)$	9	10	11	12	13	14	15	16
how many	7	13	0	5	0	0	0	1

Theorem 4.11. *If X is a smooth cubic fourfold over $\mathbb{F}_2$, then the algebraic rank $r = \operatorname{rk}\operatorname{CH}^2(X)$ can be any integer $1 \leq r \leq 10$, or $r = 12, 16$ and, furthermore, there are ordinary cubic fourfolds of every algebraic rank up to rank* 10.

Remark 4.12. In fact, the extremal cubic fourfold X_1 in Remark 3.5 is the unique smooth cubic fourfold over $\mathbb{F}_2$ with algebraic rank 16.

TABLE 7. Rank of the group of geometric cycles $\operatorname{CH}^2(\overline{X})$

$\operatorname{rk}\operatorname{CH}^2(\overline{X})$	1	3	5	7	9	11
how many	107552	254144	153410	179596	107911	98978

$\operatorname{rk}\operatorname{CH}^2(\overline{X})$	13	15	17	19	21	23
how many	61054	50777	27339	14588	5525	8688

Theorem 4.13. *If X is a smooth cubic fourfold over $\mathbb{F}_2$ then the geometric rank $r = \operatorname{rk}\operatorname{CH}^2(\overline{X})$ can be any odd number $r \leq 23$, and all such ranks $r \leq 21$ are realized by ordinary smooth cubic fourfolds.*

We remark that since the Tate conjecture holds for a supersingular cubic fourfold X, the algebraic cycles $\operatorname{CH}^2(\overline{X})$ span $H^4_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(2))$ and so any supersingular cubic fourfold has geometric rank $\operatorname{rk}\operatorname{CH}^2(\overline{X}) = 23$.

The tables below summarize our computations of the ranks of the algebraic and geometric Chow groups of smooth cubic fourfolds over $\mathbb{F}_2$.

4.7. **Implementation and verification.** We implemented Algorithm 4.3 in `C++` using many of the ideas and optimizations introduced in [17, Algorithm 15] and developed in [27, §8] and [33], including a precomputation of Galois orbit representatives of $\mathbb{F}_{2^m}$ and utilizing Intel `x86` vector carry-less finite field multiplication (`vpclmulqdq`) further reducing clock cycles. We also utilized many of the implementation ideas in [1, §5], including a precomputation of roots of quadratic and cubic polynomials. We implemented and optimized a parallelization of this algorithm over the entire database on a 24-core 48-thread 3.0 GHz cluster housed at Dartmouth, which in total, took the equivalent of a month of single thread CPU time.

In addition to the verification checks from [1, §5] (including comparing individual point counts with `Magma`'s [8] algorithm, comparing against known zeta function computations in the literature, and projecting from different lines) we also tested the algorithm on selection of mildly and highly singular cubic fourfolds. There checks can be found in [4, `/src/CubicLib/test`]

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